Interview Summary	Application No.	Applicant(s)
	10/660,117	SAVARI, SERAP AYSE
	Examiner	**Art Unit**
	Peguy JeanPierre	2819

All participants (applicant, applicant's representative, PTO personnel):

(1) *Peguy JeanPierre*. (3)_____.

(2) *K. Mason*. (4)_____.

Date of Interview: *10 February 2006*.

Type: a)☒ Telephonic b)☐ Video Conference
c)☐ Personal [copy given to: 1)☐ applicant 2)☐ applicant's representative]

Exhibit shown or demonstration conducted: d)☐ Yes e)☐ No.
If Yes, brief description: _____.

Claim(s) discussed: *none*.

Identification of prior art discussed: _____.

Agreement with respect to the claims f)☒ was reached. g)☐ was not reached. h)☐ N/A.

Substance of Interview including description of the general nature of what was agreed to if an agreement was reached, or any other comments: *Applicant agreed to send copy of an article of D. Perrin which is incorporated by reference in the specification* .

(A fuller description, if necessary, and a copy of the amendments which the examiner agreed would render the claims allowable, if available, must be attached. Also, where no copy of the amendments that would render the claims allowable is available, a summary thereof must be attached.)

THE FORMAL WRITTEN REPLY TO THE LAST OFFICE ACTION MUST INCLUDE THE SUBSTANCE OF THE INTERVIEW. (See MPEP Section 713.04). If a reply to the last Office action has already been filed, APPLICANT IS GIVEN A NON-EXTENDABLE PERIOD OF THE LONGER OF ONE MONTH OR THIRTY DAYS FROM THIS INTERVIEW DATE, OR THE MAILING DATE OF THIS INTERVIEW SUMMARY FORM, WHICHEVER IS LATER, TO FILE A STATEMENT OF THE SUBSTANCE OF THE INTERVIEW. See Summary of Record of Interview requirements on reverse side or on attached sheet.

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____
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RYAN, MASON & LEWIS, LLP
ATTORNEYS AT LAW
1300 POST ROAD, SUITE 205
FAIRFIELD, CONNECTICUT 06824
Telephone: (203) 255-6560
Facsimile: (203) 255-6570

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WORDS OVER A PARTIALLY COMMUTATIVE ALPHABET

Dominique Perrin
L.I.T.P., Universite Paris 7

INTRODUCTION

Many interesting combinatorial problems on words deal with rearrangements of words. One of the goals of such rearrangements is to provide bijective mappings between sets of words satisfying certain properties and therefore give some enumeration results on words. The interested reader may consult the Chapter by D. Foata in Lothaire's book [11] where some examples of rearrangements are developped. The algorithms involved in such rearrangements are, by the way, close to the more popular ones since many sorting problems can usefully be formulated in terms of rearrangements. The study of rearrangements has lead D. Foata to consider words over an alphabet in which some of the letters are allowed to commute. And this, in turn, could have raised the interest for studying "in abstracto" problems concerning these words and the structure of the commutation monoids which is their habitat. Nonetheless, it happened on the contrary that words on partially commutative alphabets became of interest to computer scientists studying problems of concurrency control. Roughly speaking, the alphabet considered in this framework is made of functions and the commutation between these functions corresponds to the commutation of mappings under composition. A typical problem is then to decide wether, up to the commutation rule, a given word is equivalent to one in a special form (see [13], chapter 10 for an exposition of this problem). My own interest in such questions was motivated by the work of M.P. Flé and G. Roucairol [9] who proved a surprising result on finite automata in commutation monoids motivated by problems of concurrency control. The aim of this paper is to present a survey of results obtained recently on commutation monoids including a generalization of the above mentionned. It does not intend to be a comprehensive exposition and many facets of the question have been left in the dark. The first section introduces some terminology and definitions. The second section contains the discussion of two normal form theorems in commutation monoids. The third section contains some results on the structure of commutation monoids. Finally, in the last section, I will discuss the problem of finite automata and commutation monoids.

1. FREE PARTIALLY COMMUTATIVE MONOIDS

We shall consider a finite alphabet A and a binary relation Θ on the set A. We suppose that Θ is symmetric. For two letters $a,b \in A$, we denote :

$$ab = ba$$

whenever the pair (a,b) is in relation by Θ. We consider the congruence of A^* generated by this relation and we denote

$$u \equiv v \mod.\Theta$$

if the two words u,v are congruent. This means that there exists words $w_0, w_1, \ldots, w_k$ with $w_0 = u, w_k = v$ and for each i $(0 \leq i \leq k-1)$ $w_i = r_i a b s_i$, $w_{i+1} = r_i b a s_i$ with $(a,b) \in \Theta$.

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We denote $M(A,\Theta)$ the quotient of A^* by the congruence $\equiv$. It is called the *free partially commutative monoid* generated by A with respect to the relation Θ. Such a monoid will also be called a *commutation monoid*.

A simple example of a commutation monoid is obtained by considering a direct product

$$M = A^*_1 \times A^*_2 \times \ldots \times A^*_n$$

of free monoids.

A useful characterization of equivalent words, proved in [6] is the following.

We denote by $|u|_a$ the number of occurrences of the letter a in the word u. For a subset B of A, we denote by $\pi_B(u)$ the projection of u on B^* which is obtained by erasing all letters which are not in B.

PROPOSITION 1.1. *One has*

$$u \equiv v \mod. \Theta$$

iff

(i) *for each* $a \in A$, $|u|_a = |v|_a$

(ii) *for each pair* a, b *of letters such that* $(a,b) \notin \Theta$ *one has*

$$\pi_{a,b}(u) = \pi_{a,b}(v)$$

Proof : The conditions are obviously necessary. Conversely, we can use an induction on the common length of u, v. The property is clear if u, v are letters. For the induction step, let $u = au'$, $v = bv'$ with $a, b \in A$ and $u', v' \in A^*$. If $a = b$, we are done by the induction hypothesis. We suppose $a \neq b$. Since $|u|_a = |v|_a$, we have $|v|_a \geq 1$. Let $v' = ras$ with $|r|_a = 0$. Let c be any letter occurring in br. If we had $(a,c) \notin \Theta$, then $\pi_{a,c}(u)$ would begin by a and $\pi_{a,c}(v)$ would begin by c, a contradiction. Therefore $bra \equiv abr$. Now the pair u', brs satisfies the conditions (i) and (ii) and therefore the two words are equivalent by the induction hypothesis. Hence

$$u = au' \equiv abrs \equiv bras = v. \qquad \square$$

There is a representation of the elements of the monoid $M(A,\Theta)$ by labeled directed graphs which is obtained as follows.

We shall associate to each word $u \in A^*$ a graph $G(u)$ called its *dependency graph*. Let $u = a_1 a_2 \ldots a_n$ with $a_i \in A$. The vertices of G are the integers $1,2,\ldots,n$ and integer i is labeled by the letter a_i. There is an edge (i,j) iff

(i) $i < j$
(ii) $(a_i, a_j) \notin \Theta$ or $a_i = a_j$

(iii) for each k with $i < k < j$ we have $a_j \neq a_k$

The number of edges in $G(u)$ is at most kn where $k = Card(A)$. In fact, for each j, the number of edges (i,j) is at most k since $a_i \neq a_k$ when (i,j) and (k,j) are edges.

PROPOSITION 1.2. *One has* $G(u) = G(v)$ *iff* $u \equiv v$.

Proof : If $u \equiv v$ then certainly $G(u) = G(v)$ since this implication is obvious for $u = rabs$, $v = rbas$ with $(a,b) \in \Theta$. Conversely, we use Proposition 1.1. Condition (i) is trivially satisfied. Condition (ii) is also satisfied since there is an edge between any two consecutive positions of each word $\pi_{a,b}(u)$, $\pi_{a,b}(v)$. □

A simple algorithm to compute the dependency graph of u consists in scanning u from left to right and keeping for each letter an integer indicating its last occurrence in the left factor of u that has been scanned.

An example of a commutation monoid can be obtained as follows. Let I, B be two sets. The set M of mappings

$$m : I \to B^*$$

has a structure of a monoid with a product defined by

$$mn(i) = m(i)n(i)$$

It is isomorphic to the direct product of $Card(I)$ copies of B^* and therefore is a commutation monoid.

When $I = B$, this monoid is called in [3] the *flow monoid* and its elements are called *flows*.

2. NORMAL FORMS

We present in this section two normal form theorems. The first one is due to D. Foata and the second one to D. Knuth.

We begin with Foata Normal Form. Let $M(A, \Theta)$ be a free partially commutative monoid. We suppose that the alphabet A is totally ordered.

We say that a word $w \in A^+$ is in *Foata Normal Form* if

$$w = u_1 u_2 \ldots u_n \qquad (2.1)$$

where the words $u_1, u_2, \ldots, u_n$ satisfy the following conditions :

(i) each word u_i is a non empty product of distinct letters commuting with each other in increasing order
(ii) if a is a letter of u_{i+1} not appearing in u_i then there is a letter b of u_i such that

$$(a,b) \notin \Theta.$$

Roughly speaking, a word is in normal form when all its letters have been pushed at the left as much as possible to form blocks of distinct letters.

For instance, in the free commutative monoid over $A = \{a,b\}$, the word $w = (ab)(ab)(a)(a)$ is in normal form with blocks indicated by the parenthesis.

If w is in formal form, there is a unique way of factorizing w in (2.1). In fact u_1 is the largest left factor of w composed with distinct letters that commute with each other.

The following result appears in [3] (see also [10]).

THEOREM 2.1. *For each word $u \in A^*$ there is a unique word $v \in A^*$ in normal form equivalent to u.*

Proof : The proof consists in an algorithm computing the normal form which will be commented upon later on.

Let Γ be the set of words $w \in A^*$ which are in Foata Normal Form, that is satisfying conditions (i) and (ii) above. We define a mapping from $\Gamma \times A$ into Γ denoted $(u,a) \longmapsto u.a$ in the following way. Let $u = u_1 u_2 \ldots u_n$ be the decomposition of u as in Eq.(2.1). Let i be the greatest integer $(0 \leq i \leq n)$ such that the word u_i contains the letter a or a letter b such that $(a,b) \notin \Theta$. We use the rule that $i = 0$ if no such integer exists. We then define

$$u.a = u_1 u_2 \ldots u_i v_{i+1} u_{i+2} \ldots u_n$$

where v_{i+1} is the result of inserting a in u_{i+1} to obtain a word with its letters in increasing order. When $i = n$, we just set $v_{i+1} = a$.

It is obvious that, according to the definition, we have $u.a \sim ua$. Also, it is easy to verify that $u.a \in \Gamma$. This proves the existence of a normal form for any word $w \in A^*$. Furthermore, let us extend the mapping $(u,a) \longmapsto u.a$ to a mapping $(u,v) \longmapsto u.v$ from $\Gamma \times A^* \to \Gamma$ by associativity. We have for each $u \in \Gamma$ and $(a,b) \in \Theta$ the equality

$$u.ab = u.ba$$

as readily verified from the definitions. Therefore, for any words $w, w' \in A^*$ such that $w \sim w'$ we have $1.w = 1.w'$. Moreover, if w is in normal form, we have $1.w = w$. This proves the uniqueness of the normal form since if w, w' are two words in normal form that are equivalent, we have

$$w = 1.w = 1.w' = w'. \qquad \square$$

The proof given above leads to a simple algorithm to compute the normal form of a word. The normal form of a word $w[1], w[2], \ldots, w[n]$ is computed as a sequence of blocks $s[1], s[2], \ldots, s[k]$ each block being a subset of the alphabet. At the same time, one keeps track for each letter a of the alphabet of a pointer $p[a]$ indicating the index of the rightmost block in which either a or a letter b with $(a,b) \notin \Theta$ occurs in the sequence of blocks. It is then straightforward to obtain the normal form of a word by scanning its letters from left to right. The only point to be made precise is the updating of the pointers $p[b]$ after inserting a letter a on the right. This is simply obtained by the following :

$$p[a] := p[a] + 1 \; ;$$

for each b such that $(a,b) \notin \Theta$ and $p[b] \leq p[a]$ *do* $p[b] := p[a]$;

The resulting algorithm computes the normal form of a word of length n in time $O(kn)$ where k is the number of pairs (a,b) such that $(a,b) \notin \Theta$.

It is interesting to note that there is also the possibility of computing the normal form of a word by another algorithm which operates from right to left. The algorithm is somehow simpler since it does not require the use of pointers as before.

The idea is to use a stack for each letter of the alphabet. When processing a letter a in the right-to-left scanning of w, the letter is pushed on its stack and a marker is pushed on the stack of all the letters b such that $(a,b) \notin \Theta$. The normal form is obtained by recursively "peeling" the tops off the stacks as in the following example.

EXAMPLE 2.1 Let $A = \{a,b,c,d\}$ *and let the complement* $\bar{\Theta}$ *of* Θ *be given by the graph*

$$\begin{tikzcd} a \arrow[r, dash] \arrow[d, dash] & b \arrow[d, dash] \\ c \arrow[r, dash] & d \end{tikzcd}$$

We consider the word $w = abdacbab$. The completed form of the four stacks is :

a	*		
*	b		
a	*	*	*
*	*	*	d
*	b	*	*
a	*	c	*
*	b	*	*
a	b	c	d

The normal form is obtained as :

$$(a)\ (b)\ (ad)\ (bc)\ (a)\ (b)$$ □

The algorithm can be implemented in the following way : we use for each letter a in A a stack $s[a]$ of boolean with the convention that "false" is the marker. The first part which consists in filling the stacks is the following :

```
for i:=n downto 1 do begin
        a:=w[i] ; push(true, s[a]) ;
        for each b in A such that (a,b) ∉ Θ do push
        (false, s[b])
  end ;
```

The second part which computes the normal form as a word u initialized to 1 uses a set variable B initialized to the whole alphabet A. It looks like

```
while B not empty do
        for a in B do
              if empty (s[a]) then B:=B-a else begin
                  if top (s[a]) then begin
                          u:=ua ; pop (s[a]) ;
                          for b in B and (a,b)∉ Θ do pop (s[b])
                  end
              end ;
```

The idea of this algorithm is related to Viennot's theory of "pileups" which gives an interesting geometrical model of partially commutative monoids [8]. The algorithm uses several stacks. From the point of view of automata theory, the question could be raised wether it could be performed by a finite automaton. The answer is easily seen to be negative. Consider indeed the mapping

$$\alpha : u \longmapsto v$$

assigning to each word its unique equivalent word in normal form. For $A = \{a,b\}$ with $ab \equiv ba$ we have

$$\alpha^{-1}((ab)^*) = \{u \in A^* \mid |u|_a = |u|_b\}$$

This shows that α is not a rational function since the right-hand side is not a rational language.

There is, however, something related to finite automata in all this : the set of words in normal form is a rational language. This gives the following corollary, whose idea was suggested to me by Philippe Flajolet.

<u>PROPOSITION 2.2.</u> *For each $n \geq 0$, let a_n be the number of equivalence classes mod. Θ represented by words of length n. The series $\Sigma a_n z^n$ is rational.* □

EXAMPLE 2.2. We consider again the monoid of Example 2.1. The set of words in normal form is precisely the set of words having no factor equal to one of the eight words

aad, acb, bbc, bda, ccb, cda, dcb, dda

and that do not begin by *cb* or *da*. The computation of the generating series $\Sigma \alpha_n z^n$ is complicated and will be treated in Example 2.3. □

There is another normal form theorem for commutation monoids . It consists in a characterization of the smallest lexicographic element in each congruence class. The result is due to Anisimov and Knuth [1].

THEOREM 2.2. *A word $w \in A^*$ is the minimal lexicographic element of its class mod. θ iff the following condition is satisfied : for each factorization $w = rbsat$ with $r, s, t \in A^*$, $a, b \in A$ and $a < b$ there exists a letter c of bs such that $(a, c) \notin \theta$.*

Proof : First suppose that w is minimal. Let $w = rbsat$ with $r, s, t \in A^*$ and $a, b \in A$, $a < b$. If we had $bsa \equiv abs$ then also

$$w \equiv rabst$$

but $rabst < rbsat$ whence a contradiction. Hence there exists a letter c of s that does not commute with a. This proves that w satisfies the condition of the statement.

Conversely, suppose that there is a word $u \equiv w$ with $u < w$. Let r be the longest common left factor of u, w. Then $u = rau'$, $w = rbw'$ with $a < b$. Since $|u|_a = |w|_a$ there is a factorization of w' in $w' = sat$ with $|s|_a = 0$. Then all the letters of s commute with a since, otherwise, we would not have $u \equiv v$. Therefore w does not satisfy the condition of the statement. □

An algorithm to compute the lexicographic normal form could use the dependency graph of section 1. It consists in selecting recursively the smallest letter without predecessor and deleting the corresponding vertex.

The set of words in lexicographic normal form is again a rational language as illustrated in the following example.

EXAMPLE 2.3. With the same monoid as in Example 2.1, the set of words in lexicographic normal form is the set of words having no factor equal to cb or da. This gives the generating series of the numbers α_n of classes mod. θ of words of length n

$$\alpha = \frac{1}{1 - 4z + 2z^2}$$

Indeed, let X denote the set of words in normal form. We have

$$X(1 + cb + da) = XA + 1$$

whence the above equality. □

3. STRUCTURE OF COMMUTATION MONOIDS

There is a family of partially commutative free monoids which is well-known and simple enough. It is the family of monoids which are a direct product of free monoids. The following result, due to M. Clerbout and M. Latteux [4], shows that this family of monoids gives, in a sense, the generic case.

THEOREM 3.1. *Any finitely generated commutation monoid can be embedded into a finitely generated monoid which is a direct product of free monoids.*

Proof : Let $M = M(A,\Theta)$ with A finite. Let $A' \subset A$ be the set of letters which commute with all other letters in A.

$$A' = \{a \in A \mid \forall b \in A,\ (a,b) \in \Theta\}$$

Let $A' = \{a_1, a_2, \ldots, a_k\}$ and $A'' = A - A'$. The monoid M is isomorphic with the direct product

$$M(A'',\Theta) \times a^*_1 \times a^*_2 \times \ldots \times a^*_k$$

The proof is therefore only needed for a monoid $M(A,\Theta)$ such that $A' = \emptyset$. Let $B = A \times A$ and let $\alpha : A^* \to B^*$ be the morphism defined for $a \in A$ by

$$\alpha(a) = (a,a_1)(a,a_2) \ldots (a,a_k)$$

where $\{a_1, a_2, \ldots, a_k\}$ is, in some order, the set of letters b such that $(a,b) \notin \Theta$. Since $A' = \emptyset$, the morphism α is injective. We define a relation τ on the alphabet B by

$$\tau = B \times B - \{((a,b),(b,a)) \mid a,b \in A\}$$

The monoid $M(B,\tau)$ is clearly isomorphic to the direct product of the free monoids with two generators $\{(a,b),(b,a)\}^*$ for $a,b \in A$.

To prove that the morphism α induces a isomorphism from M into $M(B,\tau)$ we need to prove that for $u,v \in A^*$.

$$u \equiv v \ \ mod.\ \Theta \iff \alpha(u) \equiv \alpha(v) \ \ mod.\ \tau$$

The direct implication follows directly from the fact that $ab \equiv ba$ implies $\alpha(ab) \equiv \alpha(ba)$ for $a,b \in A$. For the converse, we use Proposition 1.1. Let $u,v \in A^*$ be such that $\alpha(u) \equiv \alpha(v)$ *mod.* τ. *Let* $a \in A$ be a letter. Since $A' = \emptyset$ by the discussion above, there is a letter $b \in A$ such that $(a,b) \notin \Theta$. *Then*

$$|u|_a = |\alpha(u)|_{(a,b)} = |\alpha(v)|_{(a,b)} = |v|_b$$

Therefore, condition (i) is satisfied. Let now $a,b \in A$ be such that $(a,b) \notin \Theta$. Then clearly $\pi_{a,b}(u) = \pi_{a,b}(v)$ since the projections of $\alpha(u)$, $\alpha(v)$ on $\{(a,b),(b,a)\}^*$ are equal. Hence, by Proposition 1.1, we have $u \equiv v$ mod. Θ. □

The "embedding theorem" has several consequences on the structure of commutation monoids. Among them, there is the following result which was proved by R. Cori and Y. Metivier [5].

THEOREM 3.1. *For any word* $u \in A^*$ *the set*

$$C(u) = \{v \in A^* \mid uv \equiv vu \bmod. \theta\}$$

is a rational language.

Proof : Let α be a morphism from A^* into a direct product $A^*_1 x \ldots x A^*_n$. Since, in a free monoid, the commutator of an element is a cyclic submonoid, the commutator of any element of $A^*_1 x \ldots x A^*_n$ is rational. The set $C(u)$ is the inverse image by α of a rational subset and is therefore rational. □

Another results of [5] also proved independantly by C. Duboc [7] is that the image of $C(u)$ in the monoid $M(A,\theta)$ is a finitely generated submonoid. The proof is based on the fact that if the graph of the complement of θ is connected, the set $C(u)$ is a cyclic submonoid.

4. RECOGNIZABILITY IN COMMUTATION MONOIDS

We will be interested in this section in the following problem : how can one describe the *recognizable* subsets of a commutation monoid ? Recall that, given a monoid M, a subset X of M is said to be recognizable if there exists a morphism ϕ from M into a finite monoid F which saturates X, i.e. such that $\phi^{-1}\phi(X) = X$. The family of recognizable subsets of M is denoted $Rec(M)$. When M is a free monoid, the recognizable sets coincide with the rational sets, i.e. the sets described by rational expressions. In the general case, a recognizable set is always rational but the converse is not true.

In a monoid which is a direct product of free monoids, the family of recognizable sets has a simple structure : it consists of finite unions of direct products of recognizable subsets of each component. This comes from the well-known result asserting that the recognizable subsets of a direct product of two monoids are finite unions of recognizable subsets of the components (see [2] e.g.).

The situation is more complicated in general commutation monoids. We shall concentrate here on the closure properties of recognizable sets under rational operations : union, product, star.

The closure of the family of recognizable sets under union holds, in any monoid. For products, we have the following result [6] :

THEOREM 4.1. *The product* XY *of two recognizable subsets* X, Y *of a commutation monoid is recognizable.*

Proof : We use the "embedding theorem" 3.1. According to this theorem, it is enough to prove the closure under product in the case of a direct product of free monoids. Indeed if M, N are two monoids with $M \subset N$ a set $X \subset M$ which is recognizable in N is also recognizable in M. Let M be the family of all monoids in which the family of recognizable sets is closed under

product. If M and N belong to M then also $M \times N$ belongs to M. In fact, if X, Y are recognizable in $M \times N$ we have

$$X = \bigcup_{i=1}^{n} R_i \times S_i \quad , \quad Y = \bigcup_{j=1}^{m} T_j \times V_j$$

with $R_i, T_j \in Rec(M)$ and $S_i, V_j \in Rec(N)$. Then

$$XY = \bigcup_{i,j} R_i T_j \times S_i V_j$$

and therefore $XY \in Rec(M \times N)$. Since any free monoid belongs to the family M, any direct product of free monoids also belongs to M. □

The closure under the star operation presents a difficulty. In fact, it is not true in general that the star X^* of a recognizable subset X of a commutation monoid is recognizable. It is enough to consider the case of $X = \{(1,1)\}$ in the monoid $N \times N$ since

$$X^* = \{(n,n) \mid n \geq 0\}$$

which is the diagonal of $N \times N$ and which is not recognizable.

The following theorem is the best result known at present on the closure under the star operation. We present it without proof.

THEOREM 4.2. *Let $M = M(A, \theta)$ be a commutation monoid. Let X be a recognizable subset of M satisfying the following condition : for each word $x \in A^*$ representing an element of X, the restriction of the complement of θ to the set of letters appearing in x is connected. Then the set X^* is recognizable in M.*

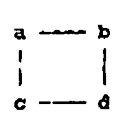
This result is due to Y. Métivier [12]. It has been obtained by successive generalizations of previous ones. First of all, the result was proved by M.P. Flé and G. Roucairol [9] under the stronger hypothesis that :

(i) X is finite

(ii) for each element x in X there is only one word in A^* representing X.

In a further work, done by R. Cori and myself [6], we were able to replace hypothesis (i) by the less restrictive hypothesis that X is recognizable. The result was also proved independantly by M. Latteux. Finally, Y. Métivier has succeeded to adapt our proof to obtain Theorem 4.2 in which condition (ii) is stated in a much weaker form.

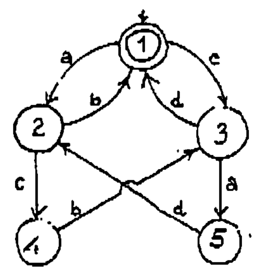
EXAMPLE 4.1. Let $A = \{a, b, c, d\}$ and θ be given by the graph of its complement by the figure below



The commutation monoid $M(A,\Theta)$ is the same as in Example 2.1. Consider the sets

$$X = \{ab, cd\}$$
$$Y = \{y \in A^* \mid \exists x \in X^*,\ y \equiv x\}$$

By Theorem 4.2, the set Y is recognizable. It is in fact recognized by the finite automaton given below



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